Section 11.1: Sequences

What is a Sequence?

Def:

Version 1

A sequence is an infinite list of numbers with a definite order.

Version 2

A sequence is a function whose domain is the positive whole numbers.

Ex 1: $1 , \frac{1}{2} , \frac{1}{4} , \frac{1}{8} , \frac{1}{16} , \frac{1}{32} , \ldots$

What is a Sequence?

<u>There are 3 ways to describe a sequence</u>

1. With a list of the first few terms

<u>Note</u>: The list should be long enough so that the pattern is obvious.

<u>Ex 1 (again)</u>: $1\ ,\ \frac{1}{2}\ ,\ \frac{1}{4}\ ,\ \frac{1}{8}\ ,\ \frac{1}{16}\ ,\ \frac{1}{32}\ ,\ \ldots$

<u>Ex 2</u>: $\frac{1}{3}\ ,\ -\frac{4}{9}\ ,\ \frac{9}{27}\ ,\ -\frac{16}{81}\ ,\ \frac{25}{243}\ ,\ \ldots$

What is a Sequence?

There are 3 ways to describe a sequence

2. With a closed form formula

Ex 3: Find the first 4 terms of each of the following sequences. Then find the 29th term and graph the sequence.

a) $a_n = (-1)^n \frac{n}{n+1}$

What is a Sequence?

There are 3 ways to describe a sequence

2. With a closed form formula

Ex 3: Find the first 4 terms of each of the following sequences. Then find the 29th term and graph the sequence.

b) $b_n = \dfrac{\sqrt{n-2}}{n!}$, $n \geq 2$

What is a Sequence?

<u>There are 3 ways to describe a sequence</u>

3. With a recursively definition

<u>Ex 4</u>: Find the first 6 terms of the following sequence…

a) $a_1 = 5$, $a_2 = -2$, and $a_n = a_{n-1}a_{n-2} - 3a_{n-1}$ for $n > 2$

Limits of Sequences

<u>Def</u>: If a_n is a sequence, then

$$\lim_{n\to\infty} a_n = L \quad \text{means...}$$

Version 1

If you go further and further to the right down the sequence list (or down the graph), the numbers in the sequence settle down to L.

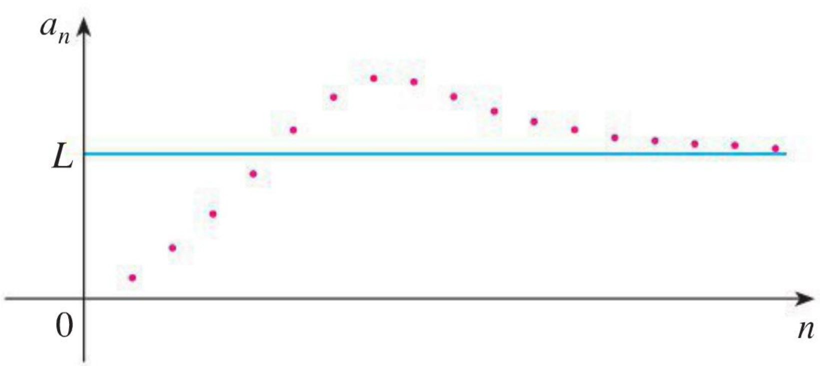
Version 2

If you plug in bigger and bigger whole numbers in for n into the formula for the sequence, the outputs settle down to the number L.

Limits of Sequences

<u>Def</u>: If a_n is a sequence and L is a real number, then

$$\lim_{n\to\infty} a_n = L \quad \text{means…}$$



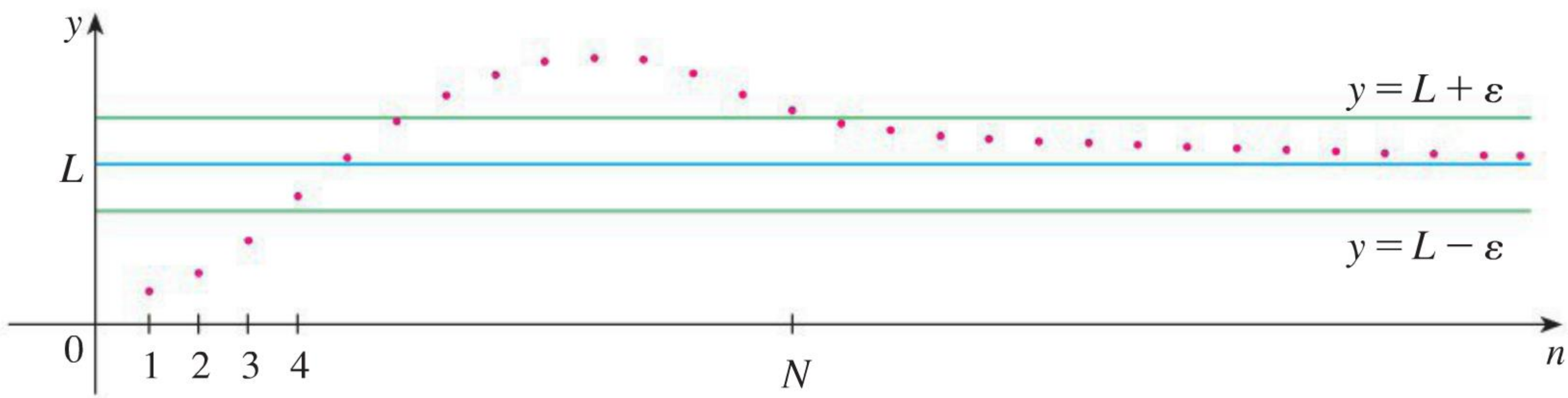
If such a real number exists, we say that the sequence a_n is a <u>convergent</u> sequence. Otherwise it is a <u>divergent</u> sequence.

Limits of Sequences

<u>Def (formal)</u>: If a_n is a sequence and L is a real number, then

$$\lim_{n\to\infty} a_n = L \quad \text{means…}$$

For every $\epsilon > 0$, there exists a number N such that
$L - \epsilon < a_n < L + \epsilon$ for all $n \geq N$.



Limits of Sequences

<u>Def (formal)</u>: If a_n is a sequence and L is a real number, then

$$\lim_{n\to\infty} a_n = L \quad \text{means…}$$

For every $\epsilon > 0$, there exists a number N such that $L - \epsilon < a_n < L + \epsilon$ for all $n \geq N$.

<u>Ex 5</u>: Is the sequence

$$\{1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \ldots\}$$

a convergent or divergent sequence?

Results That Help You Find Limits Of Sequences

Some divergent sequences

Ex 6: Find

a) $\lim\limits_{n\to\infty} (-1)^n$

b) $\lim\limits_{n\to\infty} sin\left(\frac{n\pi}{2}\right)$

Results That Help You Find Limits Of Sequences

3 Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

From Section 2.6, we had…

If $r > 0$, then $$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

So…

4 $$\lim_{n\to\infty} \frac{1}{n^r} = 0 \qquad \text{if } r > 0$$

Results That Help You Find Limits Of Sequences

$$\boxed{4} \qquad \lim_{n\to\infty} \frac{1}{n^r} = 0 \qquad \text{if } r > 0$$

Ex 7: Find

a) $\displaystyle \lim_{n\to\infty} \frac{1}{n^2} = 0$

b) $\displaystyle \lim_{n\to\infty} \frac{1}{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{n^{1/2}} = 0$

Results That Help You Find Limits Of Sequences

If $\lim_{n\to\infty} top = \#$ and $\lim_{n\to\infty} bottom = \pm\infty$, then

$$\lim_{n\to\infty} \frac{top}{bottom} = 0$$

Ex 8: Find

a) $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$

b) $\lim_{n\to\infty} \frac{1}{2^n} = 0$

Results That Help You Find Limits Of Sequences

Squeeze Theorem For Sequences

If a_n , b_n , and c_n are three sequences where …

1. $a_n \le b_n \le c_n$ eventually , and
2. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$

Then $\lim\limits_{n\to\infty} b_n = L$

Ex 9: Find

a) $\lim\limits_{n\to\infty} \dfrac{1}{n+\ln(n)}$ b) $\lim\limits_{n\to\infty} \dfrac{n!}{n^n}$

Results That Help You Find Limits Of Sequences

6 Theorem If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Ex 10: Find

a) $\lim_{n\to\infty} \dfrac{(-1)^n}{n}$

b) $\lim_{n\to\infty} \dfrac{\sin(n)}{n}$

Results That Help You Find Limits Of Sequences

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Results That Help You Find Limits Of Sequences

Ex 11: Find

a) $\lim\limits_{n\to\infty} \dfrac{12n^2}{n^2 - 5n + 1} + tan^{-1}(n)$

b) $\lim\limits_{n\to\infty} \dfrac{n}{\sqrt{4n^2 + 1}}$

Results That Help You Find Limits Of Sequences

3 Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

In conjunction with L'Hospital's Rule

Sec. 4.4: Indeterminate Forms and L'Hospital's Rule

L'Hospital's Rule

Suppose you are taking the limit of a fraction

$$\lim_{x\to a}\frac{f(x)}{g(x)}.$$

If this limit is of the type $\frac{0}{0}$ or type $\frac{\pm\infty}{\pm\infty}$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Sec. 4.4: Indeterminate Forms and L'Hospital's Rule

Other Indeterminate Forms

$$\frac{0}{0} \qquad \frac{\pm\infty}{\pm\infty} \qquad 0\cdot\infty \qquad \infty-\infty$$

$$0^0 \qquad \infty^0 \qquad 1^\infty$$

Notes

1. If the limit is of any form other than $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, use algebra to turn the limit into the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.
2. Algebra includes reciprocals, conjugates, and logs.

Results That Help You Find Limits Of Sequences

3 Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

In conjunction with L'Hospital's Rule

Ex 12: Find

a) $\lim\limits_{n\to\infty} \dfrac{\ln(n)}{\ln(2n)}$

b) $\lim\limits_{n\to\infty} n^2 e^{-n}$

Results That Help You Find Limits Of Sequences

7 Theorem If $\lim_{n \to \infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n \to \infty} f(a_n) = f(L)$$

Ex 13: Find

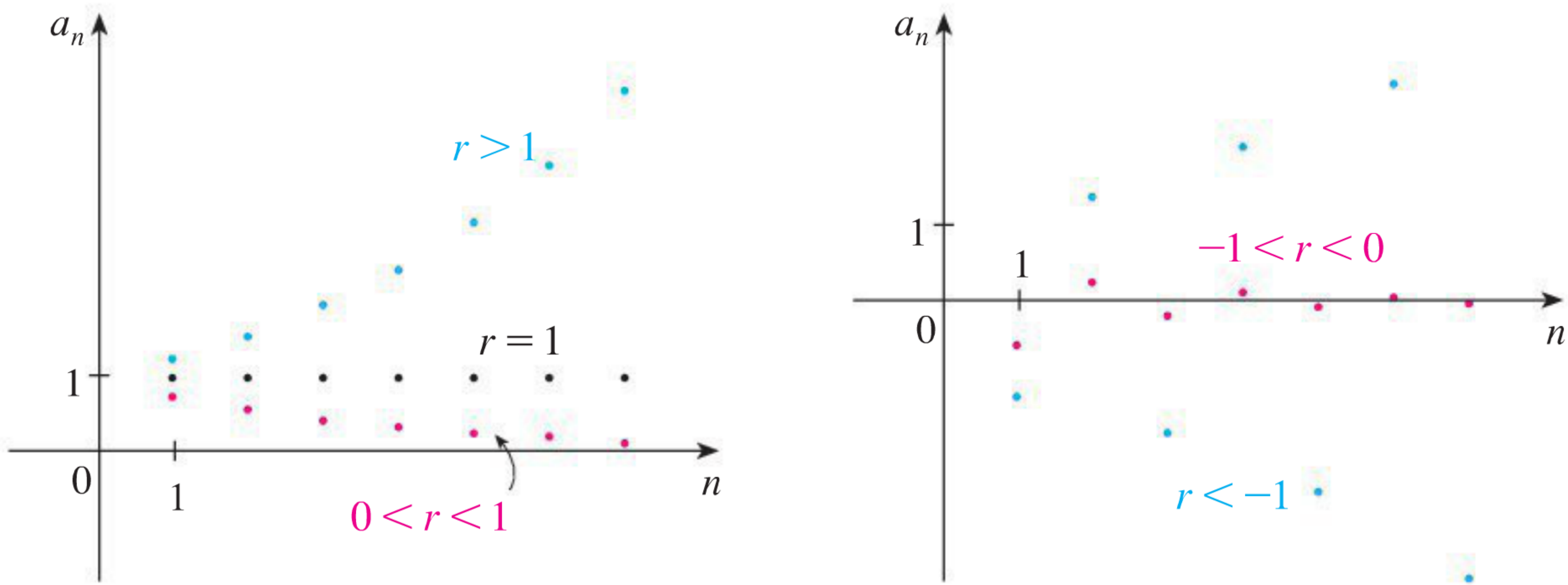
a) $\lim_{n \to \infty} sin(\pi/n)$

b) $\lim_{n \to \infty} ln(2n^2 + 1) - ln(n^2 + 1)$

Results That Help You Find Limits Of Sequences

9 The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$



Results That Help You Find Limits Of Sequences

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$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Also,

if $r > 1$, then $\displaystyle\lim_{n\to\infty} r^n = \infty$

if $r \leq -1$, then $\displaystyle\lim_{n\to\infty} r^n = DNE$

Ex 14: Find

a) $\displaystyle\lim_{n\to\infty} 4^n = \infty$

b) $\displaystyle\lim_{n\to\infty} \left(\frac{1}{3}\right)^n = 0$

Results That Help You Find Limits Of Sequences

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$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Also,

if $r > 1$, then $\lim_{n\to\infty} r^n = \infty$

if $r \leq -1$, then $\lim_{n\to\infty} r^n = DNE$

<u>Ex 14</u>: Find

c) $\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0$

d) $\lim_{n\to\infty} (-4)^n = DNE$

Results That Help You Find Limits Of Sequences

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$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Also,

if $r > 1$, then $\lim_{n\to\infty} r^n = \infty$

if $r \leq -1$, then $\lim_{n\to\infty} r^n = DNE$

<u>Ex 14</u>: Find

e) $\lim_{n\to\infty} (-1)^n = DNE$

f) $\lim_{n\to\infty} (1)^n = 1$

Results That Help You Find Limits Of Sequences

9 The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of r.

$$\lim_{n \to \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Ex 15: Find

a) $\lim\limits_{n \to \infty} 5^n 8^{-n+2}$

b) $\lim\limits_{n \to \infty} \dfrac{1}{2^n}$

Limits of Sequences

<u>Def (formal)</u>: If a_n is a sequence, then

$$\lim_{n\to\infty} a_n = \infty \quad \text{means…}$$

For every $M > 0$, there exists a number N such that $a_n > M$ for all $n \geq N$.

I.e. no matter what horizontal line $y = M$ you draw, eventually each term of the sequence is above that horizontal line.

<u>Note</u>: If $\lim_{n\to\infty} a_n = \infty$, the sequence is divergent.

Limits of Sequences

Def (formal): If a_n is a sequence, then

$$\lim_{n\to\infty} a_n = -\infty \quad \text{means…}$$

For every $M < 0$, there exists a number N such that $a_n < M$ for all $n \geq N$.

I.e. no matter what horizontal line $y = M$ you draw, eventually each term of the sequence is below that horizontal line.

Note: If $\lim_{n\to\infty} a_n = -\infty$, the sequence is divergent.

Results That Help You Find Limits Of Sequences

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Results That Help You Find Limits Of Sequences

Note: The limit laws can still be used in the case when one or more limits is $\pm\infty$ as long as the resulting calculation is an allowed calculation with infinities.

OK	NOT OK
$\infty + \infty = \infty$	$\infty - \infty = ?$
$\infty + \# = \infty$	$0 \cdot \infty = ?$

Other Indeterminate Forms

$$\frac{0}{0} \qquad \frac{\pm\infty}{\pm\infty} \qquad 0 \cdot \infty \qquad \infty - \infty$$

$$0^0 \qquad \infty^0 \qquad 1^\infty$$

Results That Help You Find Limits Of Sequences

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n$$

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$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} \quad \text{if } \lim_{n \to \infty} b_n \neq 0$$

$$\lim_{n \to \infty} a_n^p = \left[\lim_{n \to \infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Results That Help You Find Limits Of Sequences

Ex 16: Find $\displaystyle\lim_{n\to\infty} \frac{1}{n^2 + tan^{-1}(n)}$

Increasing/Decreasing/Bounded Sequences

Def: A sequence a_n is decreasing if $a_{n+1} < a_n$ for all n.

I.e. $a_1 > a_2 > a_3 > a_4 > \cdots$

I.e. If you graph the sequence, each point in the graph is lower than the previous point in the graph.

Increasing/Decreasing/Bounded Sequences

Def: A sequence a_n is increasing if $a_{n+1} > a_n$ for all n.

I.e. $a_1 < a_2 < a_3 < a_4 < \cdots$

I.e. If you graph the sequence, each point in the graph is higher than the previous point in the graph.

Increasing/Decreasing/Bounded Sequences

Def: A sequence a_n is monotonic if it is either increasing or decreasing.

Increasing/Decreasing/Bounded Sequences

Ex 17: Show that the following sequences are decreasing…

a) $a_n = \frac{3}{n+5}$

b) $b_n = \frac{n}{n^2+1}$

Increasing/Decreasing/Bounded Sequences

Def: A sequence a_n is bounded above if there exists a number M such that $a_n \leq M$ for all n.

I.e. There is some horizontal line $y = M$ such that if you graph the sequence, each point is below or on the horizontal line.

Increasing/Decreasing/Bounded Sequences

<u>Def</u>: A sequence a_n is <u>bounded below</u> if there exists a number m such that $a_n \geq m$ for all n.

I.e. There is some horizontal line $y = M$ such that if you graph the sequence, each point is above or on the horizontal line.

Increasing/Decreasing/Bounded Sequences

Def: A sequence a_n is bounded if it is bounded below and bounded above.

I.e. there exists a numbers M and m such that $m \leq a_n \leq M$ for all n.

I.e. There are 2 horizontal lines $y = m$ and $y = M$ such that if you graph the sequence, each point is between (or on) the 2 horizontal lines.

Increasing/Decreasing/Bounded Sequences

Ex 18: Which of the following sequences are bounded? If they are not bounded, are they bounded above or below?

a) $a_n = e^n$

b) $b_n = \sin(n)$

Increasing/Decreasing/Bounded Sequences

12 **Monotonic Sequence Theorem** Every bounded, monotonic sequence is convergent.

Results:

1) If a sequence is decreasing and bounded below, then it is convergent.
2) If a sequence is increasing and bounded above, then it is convergent.

Increasing/Decreasing/Bounded Sequences

Ex 19: Show that the sequence defined by

$$a_1 = 1 \qquad\qquad a_{n+1} = 1 - \frac{1}{a_n} \quad \text{for} \quad n \geq 1$$

is increasing and $a_n < 3$. Deduce that the sequence a_n is convergent and find its limit.

(Hint: Use mathematical induction)

Arithmetic and Geometric Sequences

Def: An arithmetic sequence is a sequence such that to get from any term to the next term, you keep adding the same number.

Ex: -4, 2, 8, 14, 20, 26, 32, …

Def: An geometric sequence is a sequence such that to get from any term to the next term, you keep multiplying by the same number.

Ex: $8,\ -4,\ 2,\ -1,\ \frac{1}{2},\ -\frac{1}{4},\ \frac{1}{8}, \ldots$